Factorization systems as double categories

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Plan of the presentation

1. Some double category theory,
2. Strict factorization systems $\leftrightsquigarrow$ (certain) double categories,
3. Orthogonal factorization systems $\leftrightsquigarrow$ (certain) double categories.

Definition

A *double category* X consists of objects, horizontal morphisms, vertical morphisms, and squares:

$$\begin{array}{ccc} a & \xrightarrow{g} & b \\ {\scriptstyle u}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle v} \\ c & \xrightarrow[h]{} & d \end{array}$$

The squares can be composed horizontally and vertically and both compositions are associative and unital.

It can be equivalently described as a category object in Cat, i.e. a diagram in Cat satisfying some properties:

$$X_2 \;\begin{array}{c} \xrightarrow{\;d_2\;} \\ \xrightarrow{\;d_1\;} \\ \xrightarrow{\;d_0\;} \end{array}\; X_1 \;\begin{array}{c} \xleftarrow{\;d_1\;} \\ \xrightarrow{\;s\;} \\ \xleftarrow{\;d_0\;} \end{array}\; X_0$$

Duals

A double category X admits 8 duals: the *vertical opposite* X^v, *horizontal opposite* X^h, *transpose* X^T ...
For example:

$$\begin{array}{ccc} a & \xrightarrow{g} & b \\ {\scriptstyle u}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle v} \\ c & \xrightarrow[h]{} & d \end{array} \quad \text{in } X \qquad \Longleftrightarrow \qquad \begin{array}{ccc} a & \xrightarrow{u} & c \\ {\scriptstyle g}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle h} \\ b & \xrightarrow[v]{} & d \end{array} \quad \text{in } X^T$$

Basic examples

Example

$\mathcal{C}$ a category, there is double category $\mathrm{Sq}(\mathcal{C})$ such that:

- objects are the objects of $\mathcal{C}$,
- vertical and horizontal morphisms are morphisms of $\mathcal{C}$,
- squares are commutative squares in $\mathcal{C}$

Example

We will encounter these two of its sub-double categories: $\mathrm{PbSq}(\mathcal{C}) \supseteq \mathrm{MonoPbSq}(\mathcal{C})$

Example

There is double category BOFib of (small) categories, bijections on objects, discrete opfibrations, pullback squares.

Strict factorization systems $\leftrightsquigarrow$ (certain) double categories

Strict factorization systems

Definition

A *strict factorization system* on a category $\mathcal{C}$ consists of two wide subcategories $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$ with the property that:
For every morphism $f \in \mathcal{C}$ there exist unique $e \in \mathcal{E}$, $m \in \mathcal{M}$ with:

$$f = m \circ e.$$

Definition

Denote by $\mathcal{SFS}$ the category whose:

- objects are strict factorization systems $\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}$,
- a morphism $(\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}) \to (\mathcal{E}' \subseteq \mathcal{C}' \supseteq \mathcal{M}')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ satisfying $F(\mathcal{E}) \subseteq \mathcal{E}'$ and $F(\mathcal{M}) \subseteq \mathcal{M}'$.

Example

Given categories $\mathcal{A}, \mathcal{B}$, consider $\mathcal{A} \times \mathcal{B}$ and denote:

$$\begin{aligned} \mathcal{E} &:= \{(f, 1_b) \mid & f \in \operatorname{mor} \mathcal{A}, && b \in \mathcal{B}\}, \\ \mathcal{M} &:= \{(1_a, g) \mid & g \in \operatorname{mor} \mathcal{B}, && a \in \mathcal{A}\}, \end{aligned}$$

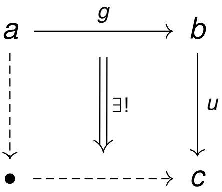
Every morphism $(f, g) \in \mathcal{A} \times \mathcal{B}$ admits a unique $(\mathcal{E}, \mathcal{M})$-factorization:

$$(f, g) = (1, g) \circ (f, 1).$$

Codomain-discrete double categories

Definition

A double category X will be called *codomain-discrete* if every top-right corner can be uniquely filled into a square:



Remark

This amounts to requiring that the codomain functor $d_0 : X_1 \to X_0$ is a discrete opfibration.

Example

If T is a very nice 2-monad on Cat, for any T-algebra (A, a), its *resolution*:

$$T^2A \;\underset{\longrightarrow\; Ta \;\longrightarrow}{\overset{\longrightarrow\; m_A \;\longrightarrow}{\longleftarrow\; Ti_A \;\longleftarrow}}\; TA$$

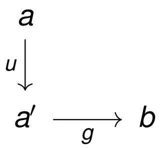
Is a double category and its transpose is codomain-discrete.

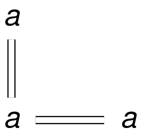
C.d. double categories ⇝ SFS' (1/2)

Construction

Let X be codomain-discrete. By the *category of corners* associated to X we mean a category $\mathrm{Cnr}(X)$ such that:

- objects are the objects of X,
- a morphism $a \to b$ is a tuple (u, g) of a vertical and a horizontal morphism in X (below left):

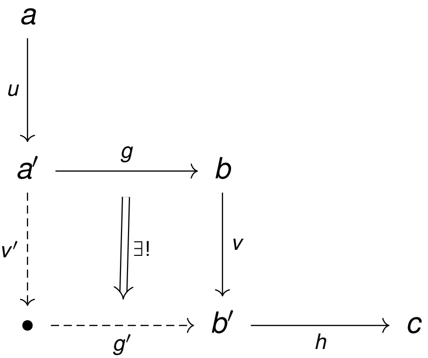




- the identity on an object a is the corner $(1_a, 1_a)$ (above right).

C.d. double categories $\rightsquigarrow$ SFS' (2/2)

The composite of $(u, g) : a \to b$ and $(v, h) : b \to c$ is defined using the unique filler square, in this case it is the corner $(v' \circ u, h \circ g') : a \to c$:



The category of corners $\mathrm{Cnr}(X)$ has two canonical wide subcategories consisting of "vertical" and "horizontal" corners:

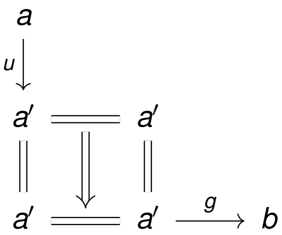
$$\mathcal{E}_X := \{(u, 1) \mid u \in \text{vmor } X\} \qquad \mathcal{M}_X := \{(1, g) \mid g \in \text{hmor } X\}.$$

Lemma

Let X be codomain-discrete. Then $(\mathcal{E}_X, \mathcal{M}_X)$ is a strict factorization system on the category $\mathrm{Cnr}(X)$.

Proof

Every corner (u, g) factors uniquely as $(1, g) \circ (u, 1)$:



SFS' $\rightsquigarrow$ c.d. double categories (1/2)

Construction

Let $(\mathcal{E}, \mathcal{M})$ be two classes of morphisms in a category $\mathcal{C}$, both closed under composition and containing all identities. Define a double category $D_{\mathcal{E},\mathcal{M}}$ as follows:

- The objects are the objects of $\mathcal{C}$,
- vertical morphisms are those of $\mathcal{E}$,
- horizontal morphisms are those of $\mathcal{M}$,
- the squares are commutative squares in $\mathcal{C}$.

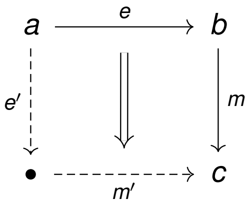
SFS' $\rightsquigarrow$ c.d. double categories (2/2)

Lemma

Let $(\mathcal{E}, \mathcal{M})$ be a strict factorization system on a category $\mathcal{C}$. Then $D_{\mathcal{E},\mathcal{M}}$ is codomain-discrete.

Proof

The unique filler square is given by the unique $(\mathcal{E}, \mathcal{M})$-factorization of the morphism $m \circ e$ in $\mathcal{C}$:



SFS' $\leftrightsquigarrow$ cod. discr. double categories

Theorem

The assignments:

$$
\begin{aligned}
(\mathcal{E}, \mathcal{M}) &\mapsto D_{\mathcal{E},\mathcal{M}}, \\
X &\mapsto (\mathcal{E}_X, \mathcal{M}_X),
\end{aligned}
$$

Are equivalence inverse to each other and thus induce an equivalence between strict factorization systems and codomain-discrete double categories.

$$
\mathcal{SFS} \underset{D}{\overset{\mathrm{Cnr}(-)}{\rightleftarrows}} \mathrm{CodDiscr} \qquad (\simeq)
$$

Orthogonal factorization systems $\leftsquigarrow\!\!\rightsquigarrow$ (certain) double categories

OFS $\leftrightsquigarrow$ (certain) double categories

The goal is to prove an analogue of the above result for orthogonal factorization systems. To do this, we need three ingredients:

1. bicartesian squares,
2. invariance,
3. the notion of a “joint monicity” of a pair of a vertical and a horizontal morphism in a double category.

Orthogonal factorization systems

Definition

An *orthogonal factorization system* $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ consists of two wide sub-categories $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$ satisfying:

- For every morphism $f \in \mathcal{C}$ there exist $e \in \mathcal{E}, m \in \mathcal{M}$ such that $f = m \circ e$, and if $f = m'e'$ is a second factorization with $e' \in \mathcal{E}, m' \in \mathcal{M}$, there exists a unique morphism θ so that this commutes:

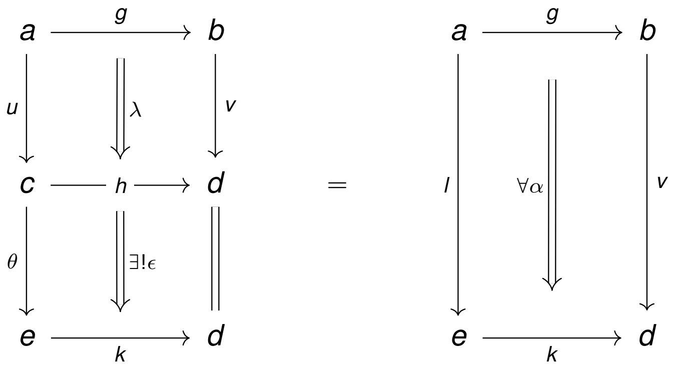
$$\begin{array}{ccccc} a & \xrightarrow{\ e\ } & a' & \xrightarrow{\ m\ } & b \\ \| & & \Big\downarrow{\scriptstyle \exists!\theta} & & \| \\ a & \xrightarrow[\ e'\ ]{} & a'' & \xrightarrow[\ m'\ ]{} & b \end{array}$$

- we have that $\mathcal{E} \cap \mathcal{M} = \{\text{isomorphisms in } \mathcal{C}\}$.

Bicrossed double categories (1/2)

Definition

A square λ in a double category X will be called *opcartesian* if it's an opcartesian morphism with respect to the codomain functor $d_0 : X_1 \to X_0$. In elementary terms:



Bicrossed double categories (2/2)

Given a double category X, denote $X^* := ((X^v)^h)^T$.

Definition - Ingredient 1

A square λ in a double category X will be called *bicartesian* if it is opcartesian in both X and X^*.

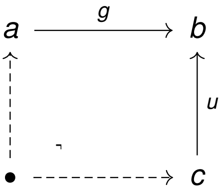
Definition

A double category X will be called *bicrossed* it every top-right corner can be filled to a (not necessarily unique) bicartesian square. Moreover, bicartesian squares are closed under horizontal and vertical compositions and identities.

Bicrossed double categories - Examples

Example

$\mathcal{C}$ a category with pullbacks, $\mathrm{Sq}(\mathcal{C})^v$, $\mathrm{PbSq}(\mathcal{C})^v$, $\mathrm{MonoPbSq}(\mathcal{C})^v$. In each of these the filler is given by a pullback square:



Example

BOFib^v. This is because both bijections on objects and discrete opfibrations are stable under pullbacks.

The category of corners

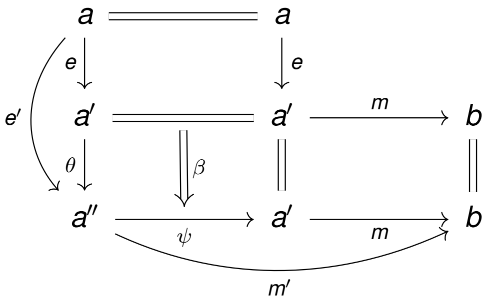
Construction

Let X be bicrossed. Assume every square is bicartesian. By the *category of corners* associated to X we mean a category $\mathrm{Cnr}(X)$ such that:

- objects are the objects of X,
- a morphism $a \to b$ is an equivalence class $[e, m]$ of tuples of a vertical morphism followed by a horizontal one in X,
- the identity on an object a is the equivalence class $[1_a, 1_a]$ (above right).

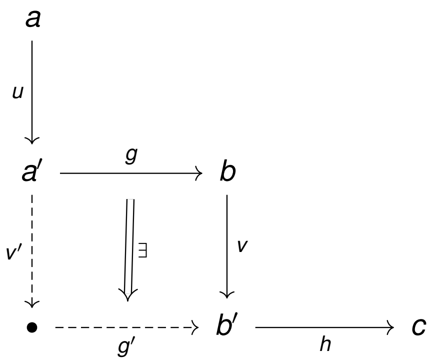
The category of corners

We consider two corners (e, m), (e', m') with the same domain and codomain equivalent if and only if there exists a square β like this:



The category of corners

The composite of $[u, g] : a \to b$ and $[v, h] : b \to c$ is defined using a **choice** of **some** bicartesian filler square, in this case it is the equivalence class $[v' \circ u, h \circ g'] : a \to c$:



The category of corners - Examples

Example

Consider $\mathsf{PbSq}(\mathcal{C})^v$ for $\mathcal{C}$ with pullbacks. $\mathsf{Cnr}(\mathsf{PbSq}(\mathcal{C})^v)$ has objects the objects of $\mathcal{C}$, while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} a & & \\ {\scriptstyle u}\uparrow & & \\ a' & \xrightarrow[g]{} & b \end{array}$$

In fact, $\mathsf{Cnr}(\mathsf{PbSq}(\mathcal{C})^v) \cong \mathsf{Span}(\mathcal{C})$.

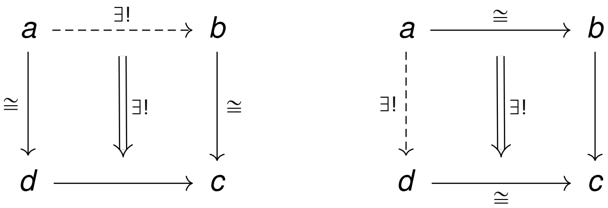
Similarly,

$$\mathsf{Cnr}(\mathsf{MonoPbSq}(\mathcal{C})^v) \cong \mathsf{Par}(\mathcal{C}),$$
$$\mathsf{Cnr}(\mathsf{BOFib}^v) \cong \mathsf{Cof}.$$

Ingredient 2

Definition - Ingredient 2

A double category X is *invariant* if the following boundaries admit a unique filler:



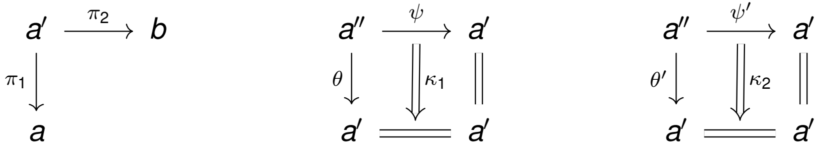
Example

All of our previous guests: $\mathrm{Sq}(\mathcal{C})$, $\mathrm{PbSq}(\mathcal{C})$, $\mathrm{MonoPbSq}(\mathcal{C})$, BOFib.

Ingredient 3

Definition - ingredient 3

A top-left corner (π_1, π_2) in a double category X is said to be *jointly monic* if, given squares κ_1, κ_2 pictured below:



We have the following implication:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\psi = \pi_2\psi') \Rightarrow (\theta = \theta', \psi = \psi').$$

Ingredient 3 - Example

Example

In Sq($\mathcal{C}$) a pair (π_1, π_2) of pullback projections is jointly monic, as this condition reduces to:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\theta = \pi_2\theta') \Rightarrow (\theta = \theta').$$

Example

In MonoPbSq($\mathcal{C}$) any pair (π_1, π_2) is jointly monic because π_1 is a monomorphism.

Example

In BOFib any pair is jointly monic. It can be proven.

Fact. double categories $\rightsquigarrow$ OFS'

Definition

A double category X is said to be a *factorization double category* if:

- every square is bicartesian and every top-right corner can be filled to a square,
- X is invariant,
- every top-left corner in X^v is jointly monic.

Let X be a factorization double category. Define the classes of "vertical" and "horizontal" corners $\mathcal{E}_X, \mathcal{M}_X$ on the category $\mathrm{Cnr}(X)$ as before. We have:

Proposition

Let X be a factorization double category. Then $(\mathcal{E}_X, \mathcal{M}_X)$ is an orthogonal factorization system on the category $\mathrm{Cnr}(X)$.

Fact. double categories $\leftsquigarrow\!\rightsquigarrow$ OFS'

Proposition

Let $(\mathcal{E}, \mathcal{M})$ be an orthogonal factorization system on a category $\mathcal{C}$. Then $D_{\mathcal{E},\mathcal{M}}$ is a factorization double category.

Theorem

The assignments are again equivalence inverse to each other and induce an equivalence:

$$\mathcal{OFS} \underset{D}{\overset{\mathrm{Cnr}(-)}{\leftrightarrows}} \mathrm{FactDbl} \quad (\simeq)$$

Examples (1/2)

Example

$\mathcal{C}$ a category with pullbacks, $\mathrm{MonoPbSq}(\mathcal{C})^v$ is a factorization double category. Thus $\mathrm{Cnr}(\mathrm{MonoPbSq}(\mathcal{C})^v) = \mathrm{Par}(\mathcal{C})$ admits an orthogonal factorization system given by "restricted identity maps" and *total maps*:

$$\begin{array}{ccc} a & & \\ \iota \big\uparrow & & \\ a' & = & a' \end{array} \qquad\qquad \begin{array}{ccc} a & & \\ \Vert & & \\ a & \xrightarrow{g} & b \end{array}$$

Examples (2/2)

Example

BOFib^v is a factorization double category and $\mathsf{Cnr}(\mathsf{BOFib}^v) = \mathsf{Cof}$ comes equipped with an orthogonal factorization system given by (the opposites of) bijections on objects followed by discrete opfibrations.

Example

If $P : \mathcal{E} \to \mathcal{B}$ is a fibration, there is a double category X_P such that:

- objects are the objects of $\mathcal{E}$,
- vertical morphisms are P-vertical morphisms,
- horizontal morphisms are P-cartesian morphisms,
- squares are commutative squares.

X_P is a factorization double category and $\mathsf{Cnr}(X_P) = \mathcal{E}$ admits an orthogonal factorization system given by P-vertical morphisms followed by P-cartesian morphisms.

References

Mark Weber (2015)
Internal algebra classifiers as codescent objects of crossed internal categories
Theory and Applications of Categories 30.50 (2015): 1713-1792.

Miloslav Štěpán (2023)
Factorization systems and double categories
arXiv preprint arXiv:2305.06714 12(3)

Thank you.